

On a conjecture of Montgomery-Vaughan on extreme values of automorphic L -functions at 1

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Abstract. In this paper, we prove a weaker form of a conjecture of Montgomery-Vaughan on extreme values of automorphic L -functions at 1.

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§ 1. Introduction

The automorphic L -functions constitute a powerful tool for studying arithmetic, algebraic or geometric objects. For squarefree integer N and even integer k , denote by $H_k^*(N)$ the set of all newforms of level N and of weight k . It is known that

$$(1.1) \quad |H_k^*(N)| = \frac{k-1}{12} \varphi(N) + O((kN)^{2/3}),$$

where $\varphi(N)$ is the Euler function and the implied constant is absolute. Let $m \geq 1$ be an integer and let $L(s, \text{sym}^m f)$ be the m th symmetric power L -function of $f \in H_k^*(N)$ normalised so that the critical strip is given by $0 < \Re s < 1$. The values of these functions at the edge of the critical strip contain information of great interest. For example, Serre [18] showed that the Sato-Tate conjecture is equivalent to $L(1 + i\tau, \text{sym}^m f) \neq 0$ for all $m \in \mathbb{N}$ and $\tau \in \mathbb{R}$. The distribution of the values $L(1, \text{sym}^m f)$ has received attention of many authors, including Goldfeld, Hoffstein & Lieman [2], Hoffstein & Lockhart [7], Luo [12], Royer [14, 15], Royer & Wu [16, 17], Cogdell & Michel [1], Habsieger & Royer [5] and Lau & Wu [10, 11]. In particular, Lau & Wu ([10], [11]) proved the following results:

(i) For every fixed integer $m \geq 1$, there are four positive constants A_m^\pm and B_m^\pm such that for any newform $f \in H_k^*(1)$, under the Great Riemann Hypothesis (GRH) for $L(s, \text{sym}^m f)$, we

have, for $k \rightarrow \infty$,

$$(1.2) \quad \{1 + o(1)\}(2B_m^- \log_2 k)^{-A_m^-} \leq L(1, \text{sym}^m f) \leq \{1 + o(1)\}(2B_m^+ \log_2 k)^{A_m^+}.$$

Here (and in the sequel) \log_j denotes the j -fold iterated logarithm. For most values of m , the constants A_m^\pm and B_m^\pm can be explicitly evaluated, for example,

$$\begin{cases} A_m^+ = m + 1, & B_m^+ = e^\gamma & (m \in \mathbb{N}), \\ A_m^- = m + 1, & B_m^- = e^\gamma \zeta(2)^{-1} & (\text{odd } m), \\ A_2^- = 1, & B_2^- = e^\gamma \zeta(2)^{-2}, \\ A_4^- = \frac{5}{4}, & B_4^- = e^\gamma B_4'^-, \end{cases}$$

where $\zeta(s)$ is the Riemann zeta-function, γ denotes the Euler constant and $B_4'^-$ is a positive constant given by a rather complicated Euler product ([10], Theorem 3).

(ii) In the opposite direction, it was shown unconditionally that for $m \in \{1, 2, 3, 4\}$ there are newforms $f_m^\pm \in H_k^*(1)$ such that for $k \rightarrow \infty$ ([10], Theorem 2),

$$(1.3) \quad \begin{cases} L(1, \text{sym}^m f_m^+) \geq \{1 + o(1)\}(B_m^+ \log_2 k)^{A_m^+}, \\ L(1, \text{sym}^m f_m^-) \leq \{1 + o(1)\}(B_m^- \log_2 k)^{-A_m^-}. \end{cases}$$

(iii) In the aim of removing GRH and closing up the gap coming from the factor 2 in (1.2) (comparing it with (1.3)), an almost all result was established. Let $\varepsilon > 0$ be an arbitrarily small positive number, $m \in \{1, 2, 3, 4\}$ and $2 \mid k$. Then there is a subset E_k^* of $H_k^*(1)$ such that $|E_k^*| \ll H_k^*(1)e^{-(\log k)^{1/2-\varepsilon}}$ and for each $f \in H_k^*(1) \setminus E_k^*$, we have, for $k \rightarrow \infty$,

$$(1.4) \quad \{1 + O(\varepsilon_k)\}(B_m^- \log_2 k)^{-A_m^-} \leq L(1, \text{sym}^m f) \leq \{1 + O(\varepsilon_k)\}(B_m^+ \log_2 k)^{A_m^+},$$

where $\varepsilon_k := (\log k)^{-\varepsilon}$ and the implied constants depend on ε only ([11], Corollary 2).

By comparing (1.3) with (1.4), the extreme values of $L(1, \text{sym}^m f)$ seem to be given by (1.3). Clearly it is interesting to investigate further the size of exceptional set E_k^* . In the case of quadratic characters L -functions, Montgomery & Vaughan [13] proposed, based on a probabilistic model, three conjectures on the size of exceptional set. The first one has been proved recently by Granville & Soundararajan [4]. As Cogdell & Michel indicated in [1], it would be interesting to try to get, as close as possible, the analogues of the conjectures of Montgomery-Vaughan for automorphic L -functions. The analogue of Montgomery-Vaughan's first conjecture for the automorphic symmetric power L -functions can be stated as follows.

Conjecture. *Let $m \geq 1$ be a fixed integer and*

$$F_k(t, \text{sym}^m) := \frac{1}{|H_k^*(1)|} \sum_{f \in H_k^*(1), L(1, \text{sym}^m f) \geq (B_m^+ t)^{A_m^+}} 1,$$

$$G_k(t, \text{sym}^m) := \frac{1}{|H_k^*(1)|} \sum_{f \in H_k^*(1), L(1, \text{sym}^m f) \leq (B_m^- t)^{-A_m^-}} 1.$$

Then there are positive constants $c_i = c_i(m)$ ($i = 1, 2$) such that for $k \rightarrow \infty$,

$$(1.5) \quad \begin{cases} e^{-c_1(\log k)/\log_2 k} \ll F_k(\log_2 k, \text{sym}^m) \ll e^{-c_2(\log k)/\log_2 k}, \\ e^{-c_1(\log k)/\log_2 k} \ll G_k(\log_2 k, \text{sym}^m) \ll e^{-c_2(\log k)/\log_2 k}. \end{cases}$$

The aim of this paper is to prove a weaker form of this conjecture for $m = 1$. In this case, we write, for simplification of notation,

$$L(s, f) = L(s, \text{sym}^1 f), \quad F_k(t) = F_k(t, \text{sym}^1), \quad G_k(t) = G_k(t, \text{sym}^1).$$

In view of the trace formula of Petersson ([8], Theorem 3.6), it is more convenient to consider the weighted arithmetic distribution function. As usual, denote by

$$\omega_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1} \|f\|}$$

the harmonic weight in modular forms theory and define the weighted arithmetic distribution functions

$$\begin{aligned} \tilde{F}_k(t) &:= \left(\sum_{f \in H_k^*(1)} \omega_f \right)^{-1} \sum_{f \in H_k^*(1), L(1, f) \geq (e^\gamma t)^2} \omega_f, \\ \tilde{G}_k(t) &:= \left(\sum_{f \in H_k^*(1)} \omega_f \right)^{-1} \sum_{f \in H_k^*(1), L(1, f) \leq (6\pi^{-2} e^\gamma t)^{-2}} \omega_f. \end{aligned}$$

By using (1.1), the classical estimate

$$(1.6) \quad \sum_{f \in H_k^*(1)} \omega_f = 1 + O(k^{-5/6})$$

and the bound of Goldfeld, Hoffstein & Lieman [2]:

$$(1.7) \quad 1/(k \log k) \ll \omega_f \ll (\log k)/k,$$

we easily see that

$$(1.8) \quad \begin{cases} \tilde{F}_k(t)/\log k \ll F_k(t) \ll \tilde{F}_k(t) \log k, \\ \tilde{G}_k(t)/\log k \ll G_k(t) \ll \tilde{G}_k(t) \log k. \end{cases}$$

This shows that in order to prove (1.5) it is sufficient to establish corresponding estimates of the same quality for $\tilde{F}_k(t)$ and $\tilde{G}_k(t)$.

Our main result is the following one.

Theorem 1. *For any $A \geq 1$ there are two positive constants $c = c(A)$ and $C = C(A)$ such that the estimate*

$$(1.9) \quad \tilde{F}_k(t) = \{1 + \Delta_k(t)\} \exp \left\{ -\frac{e^{t-\gamma_0}}{t} \left(1 + O\left(\frac{1}{t}\right) \right) \right\}$$

holds uniformly for $k \geq 16, 2 \mid k$ and $t \leq T(k)$, where γ_0 is given by (1.24) below, $|\theta| \leq 1$ and

$$(1.10) \quad \begin{cases} \Delta_k(t) := \theta e^{t-T(k)-C} (t/T(k))^{1/2} + O_A(e^{-ce^{t/5}} + (\log k)^{-A}), \\ T(k) := \log_2 k - \frac{5}{2} \log_3 k - \log_4 k - 3C. \end{cases}$$

In particular there are two positive constants c_1 and c_2 such that

$$(1.11) \quad e^{-c_1(\log k)/\{(\log_2 k)^{7/2} \log_3 k\}} \ll F_k(T(k)) \ll e^{-c_2(\log k)/\{(\log_2 k)^{7/2} \log_3 k\}}.$$

The similar estimates for $\tilde{G}_k(t)$ and $G_k(T(k))$ hold also.

Remark 1. The estimates (1.11) of Theorem 1 can be considered as a weaker form of Montgomery-Vaughan's conjecture (1.5) for $m = 1$, since $T(k) \sim \log_2 k$ as $k \rightarrow \infty$. Moreover, if we could take $T(k) = \log_2 k$ in (1.11) then (1.9) would lead to the Montgomery-Vaughan's conjecture (1.5). Hence we fail from a shift

$$\frac{5}{2} \log_3 k + \log_4 k + 3C.$$

It seems however to be rather difficult to resolve completely this conjecture. One of the main difficulties is that there are no analogues of the quadratic reciprocity law and Graham-Ringrose's estimates for short characters sums of friable moduli [3], which have been exploited by Granville & Soundararajan [4].

In order to prove Theorem 1, we need to introduce a probabilistic model as in [1]. Consider a probability space (Ω, μ) , with measure μ . Let $\mathrm{SU}(2)^\natural$ be the set of conjugacy classes of $\mathrm{SU}(2)$. The group $\mathrm{SU}(2)$ is endowed with its Haar measure μ_H and

$$\mathrm{SU}(2)^\natural = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, \pi] \right\} / \sim$$

is endowed with the Sato-Tate measure $d\mu_{\mathrm{st}}(\theta) := (2/\pi) \sin^2 \theta d\theta$, *i.e.*, the direct image of μ_H by the canonical projection $\mathrm{SU}(2) \rightarrow \mathrm{SU}(2)^\natural$. On the space (Ω, μ) , define a sequence indexed by the prime numbers, $g^\natural(\omega) = \{g_p^\natural(\omega)\}_p$ of random matrices taking values in $\mathrm{SU}(2)^\natural$, given by

$$g_p^\natural(\omega) := \begin{pmatrix} e^{i\vartheta_p(\omega)} & 0 \\ 0 & e^{-i\vartheta_p(\omega)} \end{pmatrix}^\natural.$$

We assume that each function $g_p^\natural(\omega)$ is distributed according to the Sato-Tate measure. This means that, for each integrable function $\phi : \mathrm{SU}(2)^\natural \rightarrow \mathbb{R}$, the expected value of $\phi \circ g_p^\natural$ is

$$\mathbb{E}(\phi \circ g_p^\natural) := \int_{\Omega} \phi \circ g_p^\natural(\omega) d\mu(\omega) = \int_0^\pi \phi \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) \cdot (2/\pi) \sin^2 \theta d\theta.$$

Moreover, we assume that the sequence $g^\natural(\omega)$ is made of independent random variables. This means that, for any sequence of integrable functions $\{G_p : \mathrm{SU}(2)^\natural \rightarrow \mathbb{R}\}_p$, we have

$$\begin{aligned} (1.12) \quad \mathbb{E} \left(\prod_p G_p \circ g_p^\natural \right) &:= \int_{\Omega} \prod_p G_p \circ g_p^\natural(\omega) d\mu(\omega) \\ &= \prod_p \int_{\Omega} G_p \circ g_p^\natural(\omega) d\mu(\omega) \\ &= \prod_p \int_0^\pi G_p \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) \cdot (2/\pi) \sin^2 \theta d\theta. \end{aligned}$$

Let I be the identity matrix. Then for $\Re s > \frac{1}{2}$, the random Euler product

$$L(s, g^\natural(\omega)) := \prod_p \det (I - p^{-s} g_p^\natural(\omega))^{-1} =: \prod_p L_p(s, g^\natural(\omega))$$

turns out to be absolutely convergent a.s.

Now we define our probabilistic distribution functions

$$\begin{cases} \Phi(t) := \mathrm{Prob}(\{L(1, g^\natural(\cdot)) \geq (e^\gamma t)^2\}), \\ \Psi(t) := \mathrm{Prob}(\{L(1, g^\natural(\cdot)) \leq (6\pi^{-2} e^\gamma t)^{-2}\}). \end{cases}$$

We shall prove Theorem 1 in two steps. The first one is to compare $\tilde{F}_k(t)$ with $\Phi(t)$ (resp. $\tilde{G}_k(t)$ with $\Psi(t)$).

Theorem 2. *For any $A \geq 1$ there are two positive constants $c = c(A)$ and $C = C(A)$ such that the asymptotic formulas*

$$(1.13) \quad \tilde{F}_k(t) = \Phi(t)\{1 + \Delta_k(t)\} \quad \text{and} \quad \tilde{G}_k(t) = \Psi(t)\{1 + \Delta_k(t)\}$$

hold uniformly for $k \geq 16, 2 \mid k$ and $t \leq T(k)$, where $\Delta_k(t)$ and $T(k)$ are defined by (1.10).

The second step of the proof of Theorem 1 is the evaluation of $\Phi(t)$ (resp. $\Psi(t)$). For this, we consider a truncated random Euler product

$$L(s, g^\natural(\omega); y) := \prod_{p \leq y} L_p(s, g^\natural(\omega))$$

and the corresponding distribution functions

$$\begin{cases} \Phi(t, y) := \text{Prob}(\{L(1, g^\natural(\omega); y) \geq (e^\gamma t)^2\}), \\ \Psi(t, y) := \text{Prob}(\{L(1, g^\natural(\omega); y) \leq (6\pi^{-2}e^\gamma t)^{-2}\}). \end{cases}$$

We have

$$(1.14) \quad \Phi(t) = \Phi(t, \infty) \quad \text{and} \quad \Psi(t) = \Psi(t, \infty).$$

We shall use the saddle-point method (introduced by Hildebrand & Tenenbaum [6]) to evaluate $\Phi(t, y)$ and $\Psi(t, y)$. For this, we need to introduce some notation. For $s \in \mathbb{C}$ and $y \geq 2$, define

$$(1.15) \quad E(s, y) := \mathbb{E}(L(1, g^\natural(\omega); y)^s) \quad \text{and} \quad E(s) := E(s, \infty),$$

where $\mathbb{E}(\cdot)$ denotes the expected value. We define also

$$(1.16) \quad \phi(s, y) := \log E(s, y), \quad \phi_n(s, y) := \frac{\partial^n \phi}{\partial s^n}(s, y) \quad (n \geq 0).$$

According to Lemmas 2.3 and 8.1 below, there is an absolute constant $c \geq 2$ such that for $t \geq 4 \log c$ and $y \geq ce^t$, the equation

$$(1.17) \quad \phi_1(\kappa, y) = 2(\log t + \gamma)$$

has a unique positive solution $\kappa = \kappa(t, y)$ and for each integer $J \geq 1$, there are computable constants $\gamma_0, \gamma_1, \dots, \gamma_J$ such that the asymptotic formula

$$(1.18) \quad \kappa(t, y) = e^{t-\gamma_0} \left\{ 1 + \sum_{j=1}^J \frac{\gamma_j}{t^j} + O_J \left(\frac{1}{t^{J+1}} + \frac{e^t t}{y \log y} \right) \right\}$$

holds uniformly for $t \geq 1$ and $y \geq 2e^t$, the constant γ_0 being given by (1.24) below.

Finally write $\sigma_n := \phi_n(\kappa, y)$.

Theorem 3. We have

$$\Phi(t, y) = \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2}(e^{\gamma t})^{2\kappa}} \left\{ 1 + O\left(\frac{t}{e^t}\right) \right\}$$

uniformly for $t \geq 1$ and $y \geq 2e^t$.

Theorem 4. For each integer $J \geq 1$, we have

$$(1.19) \quad \Phi(t, y) = \exp \left\{ -\kappa \left[\sum_{j=1}^J \frac{a_j}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right] \right\}$$

uniformly for $t \geq 1$ and $y \geq 2e^t$, where the error term $R_J(\kappa, y)$ is given by

$$(1.20) \quad R_J(\kappa, y) := \frac{1}{(\log \kappa)^{J+1}} + \frac{\kappa}{y \log y}$$

and

$$(1.21) \quad a_j := \int_0^\infty \left(\frac{h(u)}{u} \right)' (\log u)^{j-1} du$$

with

$$(1.22) \quad h(u) := \begin{cases} \log \left(\frac{2}{\pi} \int_0^\pi e^{2u \cos \theta} \sin^2 \theta d\theta \right) & \text{if } 0 \leq u < 1, \\ \log \left(\frac{2}{\pi} \int_0^\pi e^{2u \cos \theta} \sin^2 \theta d\theta \right) - 2u & \text{if } u \geq 1. \end{cases}$$

As a corollary of Theorem 4, we can obtain an asymptotic developpment for $\log \Phi(t, y)$ in t^{-1} . In particular we see that the probabilistic distribution function $\Phi(t)$ decays double exponentially as $t \rightarrow \infty$.

Corollary 5. For each integer $J \geq 1$, there are computable constants a_1^*, \dots, a_J^* such that the asymptotic formula

$$(1.23) \quad \Phi(t, y) = \exp \left\{ -e^{t-\gamma_0} \left[\sum_{j=1}^J \frac{a_j^*}{t^j} + O_J(R_J(e^t, y)) \right] \right\}$$

holds uniformly for $t \geq 1$ and $y \geq 2e^t$. Further we have

$$(1.24) \quad \gamma_0 := \frac{1}{2} \int_0^\infty \frac{h'(u)}{u} du, \quad a_1^* := 1, \quad a_2^* := \gamma_0 - \frac{\gamma_0^2}{2} - \int_0^\infty \frac{h(u)}{u^2} (\log u) du.$$

In particular for each integer $J \geq 1$, we have

$$(1.25) \quad \Phi(t) = \exp \left\{ -e^{t-\gamma_0} \left[\sum_{j=1}^J \frac{a_j^*}{t^j} + O_J\left(\frac{1}{t^{J+1}}\right) \right] \right\}$$

uniformly for $t \geq 1$.

Remark 2. (i) The same results hold also for $\Psi(t, y)$.

(ii) Taking $t = \log_2 k$ and $J = 1$ in (1.25) of Corollary 5, we see that the probabilistic distribution function $\Phi(t)$ (resp. $\Psi(t)$) verifies Montgomery-Vaughan's conjecture (1.5). But (1.13) is too weak to derive this conjecture for $F_k(t)$ (resp. $G_k(t)$). This means that we must take $T(k) = \log_2 k$ in Theorem 2, which seems to be rather difficult.

(iii) Our method can be generalized (with a little extra effort) to prove that Theorems 1 and 2 hold for $L(1, \text{sym}^m f)$ for $m \geq 1$ (unconditionally when $m = 1, 2, 3, 4$ and under Cogdell-Michel's hypothesis $\text{Sym}^m(f)$ and $\text{LSZ}^m(1)$ [1] when $m \geq 5$) and that Theorems 3, 4 and Corollary 5 are true for $L(1, \text{sym}^m g^{\mathfrak{h}}(\omega); y)$ when $m \geq 1$.

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§ 2. Expression of $E(s, y)$ and existence of saddle-point

The aim of this section is to prove the existence of the saddle-point $\kappa(t, y)$, defined by equation (1.17). The first step is to give an explicit expression of $E(s, y)$, which is (1.24) of [1]. For the convenience of readers, we state it here as a lemma.

Lemma 2.1. *For prime p , real θ and complex number s , we define*

$$(2.1) \quad D_p(\theta) := \prod_{0 \leq j \leq 1} (1 - e^{i(1-2j)\theta} p^{-1})^{-1} \quad \text{and} \quad E_p(s) := \frac{2}{\pi} \int_0^\pi D_p(\theta)^s \sin^2 \theta \, d\theta.$$

Then for all $s \in \mathbb{C}$ and $y \geq 2$, we have

$$(2.2) \quad E(s, y) = \prod_{p \leq y} E_p(s).$$

Proof. Taking

$$G_p(M^{\mathfrak{h}}) = \begin{cases} \det(I - p^{-s'} M^{\mathfrak{h}})^{-s} & \text{if } p \leq y \\ 1 & \text{otherwise} \end{cases}$$

in (1.12), we get

$$\begin{aligned} \mathbb{E}(L(s', g^{\mathfrak{h}}(\omega); y)^s) &= \prod_{p \leq y} \mathbb{E}(L_p(s', g_p^{\mathfrak{h}}(\omega))^s) \\ &= \prod_{p \leq y} \int_{\Omega} \det(1 - p^{-s'} g_p^{\mathfrak{h}}(\omega))^{-s} \, d\mu(\omega) \\ &= \prod_{p \leq y} \frac{2}{\pi} \int_0^\pi (1 - 2p^{-s'} \cos \theta + p^{-2s'})^{-s} \sin^2 \theta \, d\theta. \end{aligned}$$

Taking $s' = 1$ and noticing (1.15) and (2.1), we get the desired result. \square

Lemma 2.2. *For all p and $\sigma > 0$, we have*

$$E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 > 0.$$

In particular for all $\sigma > 0$ and $y \geq 2$, we have $\phi_2(\sigma, y) > 0$.

Proof. By using the definition (2.1) of $E_p(\sigma)$, it is easy to see that

$$\begin{aligned} E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 &= \frac{4}{\pi^2} \int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta) \sin^2 \theta \, d\theta \int_0^\pi D_p(\theta)^\sigma \sin^2 \theta \, d\theta \\ &\quad - \left(\frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log D_p(\theta) \sin^2 \theta \, d\theta \right)^2 \\ &= \frac{4}{\pi^2} \int_0^\pi \int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma (\log^2 D_p(\theta_1) - \log D_p(\theta_1) \log D_p(\theta_2)) \times \\ &\quad \times \sin^2 \theta_1 \sin^2 \theta_2 \, d\theta_1 \, d\theta_2. \end{aligned}$$

In view of the symmetry in θ_1 and θ_2 , the same formula holds if we exchange the roles of θ_1 and θ_2 . Thus it follows that

$$E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 = \frac{2}{\pi^2} \int_0^\pi \int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma \log^2 \left(\frac{D_p(\theta_1)}{D_p(\theta_2)} \right) \sin^2 \theta_1 \sin^2 \theta_2 \, d\theta_1 \, d\theta_2.$$

This proves the first assertion and the second follows immediately. \square

Lemma 2.3. *There is an absolute constant $c \geq 2$ such that for $t \geq 4 \log c$ and $y \geq ce^t$, the equation $\phi_1(\sigma, y) = 2(\log t + \gamma)$ has a unique positive solution in σ . Denoting by $\kappa(t, y)$ this solution, we have $\kappa(t, y) \asymp e^t$ uniformly for $t \geq 4 \log c$ and $y \geq ce^t$.*

Proof. According to Lemma 4.3 below with the choice of $J = 1$, we have

$$\phi_1(\sigma, y) = 2(\log_2 \sigma + \gamma) + O(1/\log \sigma)$$

for $y \geq \sigma \geq 2$. Thus

$$\begin{aligned} \phi(ce^t, y) &= 2 \log(t + \log c) + 2\gamma + O\left(\frac{1}{t + \log c}\right) \\ &> 2 \log t + 2\gamma \end{aligned}$$

and

$$\begin{aligned} \phi(c^{-1}e^t, y) &= 2 \log(t - \log c) + 2\gamma + O\left(\frac{1}{t - \log c}\right) \\ &< 2 \log t + 2\gamma, \end{aligned}$$

provided that c is a large constant and $t \geq 4 \log c$. On the other hand, in view of Lemma 2.2, we know that for any $y \geq 2$, $\phi_1(\sigma, y)$ is an increasing function of σ in $(0, \infty)$. Hence the equation $\phi_1(\sigma, y) = 2(\log t + \gamma)$ has a unique positive solution $\kappa(t, y)$ and $c^{-1}e^t \leq \kappa(t, y) \leq ce^t$ for $t \geq 4 \log c$ and $y \geq ce^t$. This completes the proof. \square

§ 3. Preliminary lemmas

This section is devoted to establish some preliminary lemmas, which will be useful later.

Lemma 3.1. *Let $j \geq 0$ be a fixed real number. Then we have*

$$(3.1) \quad \int_0^\pi e^{2u \cos \theta} (1 - \cos \theta)^j \sin^2 \theta \, d\theta \asymp_j e^{2u} u^{-(j+3/2)} \quad (u \geq 1).$$

The implied constant depends on j only.

Proof. First we write

$$\begin{aligned} \int_0^\pi e^{2u \cos \theta} (1 - \cos \theta)^j \sin^2 \theta \, d\theta &= \int_0^{\pi/2} (e^{2u \cos \theta} (1 - \cos \theta)^j + e^{-2u \cos \theta} (1 + \cos \theta)^j) \sin^2 \theta \, d\theta \\ &= \int_0^1 (e^{2ut} (1-t)^j + e^{-2ut} (1+t)^j) (1-t^2)^{1/2} \, dt \\ &\asymp \int_0^1 e^{2ut} (1-t)^{j+1/2} \, dt + \int_0^1 e^{-2ut} (1-t)^{1/2} \, dt. \end{aligned}$$

By the change of variables $u(1-t) = v$, it follows that

$$\begin{aligned} \int_0^1 e^{2ut} (1-t)^{j+1/2} \, dt &= e^{2u} u^{-(j+3/2)} \int_0^u e^{-2v} v^{j+1/2} \, dv \\ &\asymp e^{2u} u^{-(j+3/2)}, \\ \int_0^1 e^{-2ut} (1-t)^{1/2} \, dt &\leq \int_0^1 e^{-2ut} \, dt \ll u^{-1}. \end{aligned}$$

We obtain the desired result by insertion of these estimates into the preceding relation. \square

Lemma 3.2. *Let $j \geq 0$ be an integer and*

$$(3.2) \quad E_{p,j}(\sigma) := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma (1 - \cos \theta)^j \sin^2 \theta \, d\theta.$$

(In particular $E_{p,0}(\sigma) = E_p(\sigma)$.) Then we have

$$E_{p,j}(\sigma) = \frac{2^{j+3}}{\pi} \int_0^1 \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1/2} (1-u)^{1/2} \, du$$

and the estimate

$$(3.3) \quad E_{p,j}(\sigma)/E_p(\sigma) \ll (p/\sigma)^j$$

holds uniformly for all primes p and $\sigma > 0$. Further if $p \geq \sigma \geq 0$, we have

$$(3.4) \quad E_p(\sigma) \asymp 1.$$

The implied constant in (3.3) depends on j only and the one in (3.4) is absolute.

Proof. By the change of variables $u = \sin^2(\theta/2)$, a simple computation shows that the first assertion is true. Obviously (3.3) holds for $j = 0$.

Now assume that it is true for j . An integration by parts leads to

$$\begin{aligned} E_p(\sigma) &\gg_j \left(\frac{\sigma}{p}\right)^j \int_0^1 \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1/2} (1-u)^{1/2} du \\ &\gg_j \left(\frac{\sigma}{p}\right)^j \int_0^1 \left\{ \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-1} \frac{4\sigma}{p} + \frac{1}{2(1-u)} \right\} \times \\ &\quad \times \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1+1/2} (1-u)^{1/2} du. \end{aligned}$$

On the other hand, we have

$$0 < u < 1 \Rightarrow \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-1} \frac{4\sigma}{p} + \frac{1}{2(1-u)} \geq \left(1 + \frac{1}{p}\right)^{-2} \frac{4\sigma}{p} \geq \frac{16\sigma}{9p}.$$

Inserting it into the preceding estimate, we see that

$$\begin{aligned} E_p(\sigma) &\gg_j \left(\frac{\sigma}{p}\right)^{j+1} \int_0^1 \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1+1/2} (1-u)^{1/2} du \\ &\asymp_j \left(\frac{\sigma}{p}\right)^{j+1} E_{p,j+1}(\sigma). \end{aligned}$$

Thus (3.3) holds also for $j+1$.

Since $(1+1/p)^{-2} \leq D_p(\theta) \leq (1-1/p)^{-2}$ for all primes p and any $\theta \in \mathbb{R}$, we have $D_p(\theta)^\sigma \asymp 1$ uniformly for $p \geq \sigma \geq 0$ and $\theta \in \mathbb{R}$. This implies (3.4). \square

Introduce the function

$$(3.5) \quad g(u) := \log \left(\frac{2}{\pi} \int_0^\pi e^{2u \cos \theta} \sin^2 \theta d\theta \right) \quad (u \geq 0)$$

and let $h(u)$ be defined as in (1.22). Clearly we have

$$(3.6) \quad h(u) = \begin{cases} g(u) & \text{if } 0 \leq u < 1, \\ g(u) - 2u & \text{if } u \geq 1, \end{cases}$$

$$(3.7) \quad h'(u) = \begin{cases} g'(u) & \text{if } 0 \leq u < 1, \\ g'(u) - 2 & \text{if } u \geq 1, \end{cases}$$

$$(3.8) \quad h''(u) = g''(u) \quad (u \geq 0, u \neq 1).$$

Lemma 3.3. *We have*

$$(3.9) \quad h(u) \asymp \begin{cases} u^2 & \text{if } 0 \leq u < 1, \\ \log(2u) & \text{if } u \geq 1, \end{cases}$$

$$(3.10) \quad h'(u) \asymp \begin{cases} u & \text{if } 0 \leq u < 1, \\ u^{-1} & \text{if } u \geq 1, \end{cases}$$

$$(3.11) \quad h''(u) \asymp \begin{cases} 1 & \text{if } 0 \leq u < 1, \\ u^{-2} & \text{if } u \geq 1, \end{cases}$$

$$(3.12) \quad h'''(u) \asymp \begin{cases} u & \text{if } 0 \leq u < 1, \\ u^{-3} & \text{if } u \geq 1. \end{cases}$$

Proof. When $0 \leq u < 1$, we have

$$e^{2u \cos \theta} = \sum_{n=0}^{\infty} \frac{(u \cos \theta)^n}{n!}.$$

From this we deduce that

$$\begin{aligned} (3.13) \quad h(u) &= \log \left(\frac{2}{\pi} \sum_{n=0}^{\infty} \frac{u^n}{n!} \int_0^{\pi} (\cos \theta)^n \sin^2 \theta \, d\theta \right) \\ &= \log \left(1 + \sum_{\ell=1}^{\infty} \frac{2 \cdot (2\ell-1)!!}{(2\ell)!(2\ell+2)!!} u^{2\ell} \right), \end{aligned}$$

where we have used the following facts:

$$\int_0^{\pi} (\cos \theta)^{2\ell+1} \sin^2 \theta \, d\theta = 0$$

and

$$\frac{2}{\pi} \int_0^{\pi} (\cos \theta)^{2\ell} \sin^2 \theta \, d\theta = \begin{cases} 1 & \text{if } \ell = 0, \\ 2 \frac{(2\ell-1)!!}{(2\ell+2)!!} & \text{if } \ell \geq 1 \end{cases}$$

and where $n!!$ denotes the product of all positive integer from 1 to n having same parity than n . Now we easily deduce, from (3.13), the desired results (3.9)–(3.12) in the case of $0 \leq u < 1$.

The estimates of (3.9)–(3.12) for $u > 1$ are simple consequences of (3.1), by noticing the following relations

$$\begin{aligned} h'(u) &= -2 \frac{\int_0^{\pi} e^{2u \cos \theta} (1 - \cos \theta) \sin^2 \theta \, d\theta}{\int_0^{\pi} e^{2u \cos \theta} \sin^2 \theta \, d\theta}, \\ h''(u) &= 4 \frac{\int_0^{\pi} e^{2u \cos \theta} (1 - \cos \theta)^2 \sin^2 \theta \, d\theta}{\int_0^{\pi} e^{2u \cos \theta} \sin^2 \theta \, d\theta} - 4 \left(\frac{\int_0^{\pi} e^{2u \cos \theta} (1 - \cos \theta) \sin^2 \theta \, d\theta}{\int_0^{\pi} e^{2u \cos \theta} \sin^2 \theta \, d\theta} \right)^2. \end{aligned}$$

This completes the proof. \square

§ 4. Estimates of $\phi_n(\sigma, y)$

The aim of this section is to prove some estimates of $\phi_n(\sigma, y)$ for $n = 0, 1, 2, 3, 4$.

Lemma 4.1. *For any fixed integer $J \geq 1$, we have*

$$(4.1) \quad \phi_0(\sigma, y) = \sigma \left\{ 2 \log_2 \sigma + 2\gamma + \sum_{j=1}^J \frac{b_{j,0}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}$$

uniformly for $y \geq \sigma \geq 3$, where $R_J(\sigma, y)$ is defined as in (1.20) and

$$(4.2) \quad b_{j,0} := \int_0^{\infty} \frac{h(u)}{u^2} (\log u)^{j-1} \, du.$$

Proof. By the definition (2.1) of $D_p(\theta)$ and the one of $E_p(\sigma)$, it is easy to see that for $p \geq \sigma^{1/2}$, we have

$$(4.3) \quad D_p(\theta)^\sigma = e^{2(\sigma/p) \cos \theta} \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\},$$

$$(4.4) \quad E_p(\sigma) = \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p) \cos \theta} \sin^2 \theta \, d\theta.$$

From these, we deduce that

$$(4.5) \quad \sum_{\sigma^{1/2} < p \leq y} \log E_p(\sigma) = \sum_{\sigma^{1/2} < p \leq y} g(\sigma/p) + O(\sigma^{1/2}/\log \sigma)$$

where $g(u)$ is defined as in (3.5).

In order to treat the sum over $p \leq \sigma$, we write

$$E_p(\sigma) = (1 - 1/p)^{-2\sigma} E_p^*(\sigma),$$

where

$$E_p^*(\sigma) := \frac{2}{\pi} \int_0^\pi \left\{ 1 + \frac{2(1 - \cos \theta)}{p} \left(1 - \frac{1}{p}\right)^{-2} \right\}^{-\sigma} \sin^2 \theta \, d\theta.$$

By using the change of variables $u = \sin^2(\theta/2)$, we have

$$\begin{aligned} E_p^*(\sigma) &= \frac{8}{\pi} \int_0^\pi \left\{ 1 + \frac{4}{p} \left(1 - \frac{1}{p}\right)^{-2} \sin^2(\theta/2) \right\}^{-\sigma} \sin^2(\theta/2) \cos^2(\theta/2) \, d\theta \\ &\geq \frac{8}{\pi} \int_0^{p/2\sigma} \left\{ 1 + \frac{4}{p} \left(1 - \frac{1}{p}\right)^{-2} u \right\}^{-\sigma} \sqrt{u(1-u)} \, du \\ &\geq \frac{8}{\pi} \left(1 + \frac{8}{\sigma}\right)^{-\sigma} \int_0^{p/2\sigma} \sqrt{u(1-u)} \, du \\ &\geq C \left(\frac{p}{\sigma}\right)^{3/2}, \end{aligned}$$

where $C > 0$ is a constant. On the other hand, we have trivially $E_p^*(\sigma) \leq 1$ for all p and $\sigma > 0$.

Thus $|\log E_p^*(\sigma)| \ll \log(\sigma/p)$ for $p \leq \sigma^{1/2}$ and

$$(4.6) \quad \sum_{p \leq \sigma^{1/2}} |\log E_p^*(\sigma)| \ll \sum_{p \leq \sigma^{1/2}} \log(\sigma/p) \ll \sigma^{1/2}.$$

Combining (4.5) and (4.6), we can write

$$\sum_{p \leq y} \log E_p(\sigma) = 2\sigma \sum_{p \leq \sigma^{1/2}} \log(1 - 1/p)^{-1} + \sum_{\sigma^{1/2} < p \leq y} g(\sigma/p) + O(\sigma^{1/2}).$$

In view of (3.6) and the following estimate

$$\sum_{\sigma^{1/2} < p \leq \sigma} (2\sigma \log(1 - 1/p)^{-1} - 2\sigma/p) \ll \sum_{\sigma^{1/2} < p \leq \sigma} \sigma/p^2 \ll \sigma^{1/2}/\log \sigma,$$

the preceding estimate can be written as

$$(4.7) \quad \sum_{p \leq y} \log E_p(\sigma) = 2\sigma \sum_{p \leq \sigma} \log(1 - 1/p)^{-1} + \sum_{\sigma^{1/2} < p \leq y} h(\sigma/p) + O(\sigma^{1/2}).$$

By using the prime number theorem in the form

$$(4.8) \quad \pi(t) := \sum_{p \leq t} 1 = \int_2^t \frac{dv}{\log v} + O\left(te^{-8\sqrt{\log t}}\right),$$

it follows that

$$(4.9) \quad \sum_{\sigma^{1/2} < p \leq y} h\left(\frac{\sigma}{p}\right) = \int_{\sigma^{1/2}}^y \frac{h(\sigma/t)}{\log t} dt + O(R_0),$$

where

$$\begin{aligned} R_0 &:= h\left(\frac{\sigma}{y}\right)ye^{-8\sqrt{\log y}} + h(\sigma^{1/2})\sigma^{1/2}e^{-4\sqrt{\log \sigma}} + \int_{\sigma^{1/2}}^y (\sigma/t)|h'(\sigma/t)|e^{-8\sqrt{\log t}} dt \\ &\ll \frac{\sigma^2}{y}e^{-8\sqrt{\log y}} + \sigma^{1/2}e^{-2\sqrt{\log \sigma}} + \int_{\sigma^{1/2}}^{\sigma} e^{-2\sqrt{\log t}} dt + \sigma^2 \int_{\sigma}^y \frac{e^{-8\sqrt{\log t}}}{t^2} dt \\ &\ll \sigma e^{-\sqrt{\log \sigma}} \end{aligned}$$

by use of Lemma 3.3.

In order to evaluate the integral of (4.9), we use the change of variables $u = \sigma/t$ to write

$$\begin{aligned} \int_{\sigma^{1/2}}^y \frac{h(\sigma/t)}{\log t} dt &= \sigma \int_{\sigma/y}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)} du \\ &= \sigma \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)} du + O(R'_0) \end{aligned}$$

where

$$\begin{aligned} R'_0 &:= \sigma \int_0^{\sigma/y} \frac{|h(u)|}{u^2 \log(\sigma/u)} du + \sigma \int_0^{\sigma^{-1/2}} \frac{|h(u)|}{u^2 \log(\sigma/u)} du \\ &\ll \frac{\sigma^2}{y \log y} + \frac{\sigma^{1/2}}{\log \sigma}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)} du &= \frac{1}{\log \sigma} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2(1 - (\log u)/\log \sigma)} du \\ &= \sum_{j=1}^J \frac{1}{(\log \sigma)^j} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2} (\log u)^{j-1} du + O\left(\frac{1}{(\log \sigma)^{J+1}}\right). \end{aligned}$$

Extending the interval of integration $[\sigma^{-1/2}, \sigma^{1/2}]$ to $(0, \infty)$ and bounding the contributions of $(0, \sigma^{-1/2}]$ and $[\sigma^{1/2}, \infty)$ by using (3.9) of Lemma 3.3, we have

$$\int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2} (\log u)^{j-1} du = b_{j,0} + O\left(\frac{(\log \sigma)^j}{\sigma^{1/2}}\right).$$

Combining these estimates, we find that

$$(4.10) \quad \sum_{\sigma^{1/2} < p \leq y} h\left(\frac{\sigma}{p}\right) = \sigma \left\{ \sum_{j=1}^J \frac{b_{j,0}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}.$$

Now the desired result follows from (4.7), (4.10) and the prime number theorem in the form

$$(4.11) \quad \sum_{p \leq \sigma} \log(1 - 1/p)^{-1} = \log_2 \sigma + \gamma + O\left(e^{-2\sqrt{\log \sigma}}\right).$$

This completes the proof. \square

Remark 3. In view of (1.3), we can write (4.1) as

$$\phi_0(\sigma, y) = \sigma \left\{ \log(B_1^+ \log \sigma)^{A_1^+} + \sum_{j=1}^J \frac{b_{j,0}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}$$

uniformly for $y \geq \sigma \geq 3$. In the case $\sigma < 0$, a similar asymptotic formula (with A_1^-, B_1^- and corresponding $b_{j,0}^-$ in place of A_1^+, B_1^+ and $b_{j,0}$) can be established uniformly for $y \geq -\sigma \geq 3$. As indicated in the introduction, Lemma 4.1 can be easily generalised to the general case $m \geq 1$. Thus we give an improvement and generalisation of Corollaries A and C of [15], of Theorem B of [5], and an improvement of Theorem 1.12 of [1]. It is worthy to indicate that our method seems to be simpler and more natural.

Lemma 4.2. *We have*

$$(4.12) \quad \frac{E'_p(\sigma)}{E_p(\sigma)} = \begin{cases} \log D_p(0) + O\left(\frac{1}{\sigma}\right) & \text{for all } p \text{ and } \sigma > 0, \\ \frac{1}{2}g'\left(\frac{\sigma}{p}\right) \log D_p(0) + O\left(\frac{1}{p^2} + \frac{\sigma}{p^3}\right) & \text{if } p \geq \sigma^{1/2}, \end{cases}$$

where $g(u)$ is defined as in (3.5).

Proof. First we write

$$(4.13) \quad \begin{aligned} E'_p(\sigma) &= \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log D_p(\theta) \sin^2 \theta \, d\theta \\ &= E_p(\sigma) \log D_p(0) + R', \end{aligned}$$

where

$$(4.14) \quad R' := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log \left(\frac{D_p(\theta)}{D_p(0)} \right) \sin^2 \theta \, d\theta.$$

Since

$$\left| \log \left(\frac{D_p(\theta)}{D_p(0)} \right) \right| = \left| -\log \left(1 + \frac{2p(1 - \cos \theta)}{(p-1)^2} \right) \right| \leq \frac{2p(1 - \cos \theta)}{(p-1)^2} \leq \frac{8(1 - \cos \theta)}{p},$$

it follows from (3.3) of Lemma 3.2 with $j = 1$ that

$$\frac{R'}{E_p(\sigma)} \ll \frac{E_{p,1}(\sigma)}{pE_p(\sigma)} \ll \frac{1}{\sigma}$$

for all p and $\sigma > 0$. This implies, via (4.13), the first estimate of (4.12).

We have

$$\begin{aligned} \log D_p(\theta) &= (\cos \theta)(2/p) + O(1/p^2) \\ &= (\cos \theta) \log D_p(0) + O(1/p^2). \end{aligned}$$

Inserting it and (4.3) into the first relation of (4.13) and in view of (4.4), we can write, for $p \geq \sigma^{1/2}$,

$$\begin{aligned} E_p'(\sigma) &= \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p) \cos \theta} \left\{ (\cos \theta) \log D_p(0) + O\left(\frac{1}{p^2}\right) \right\} \sin^2 \theta \, d\theta \\ &= \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p) \cos \theta} (\cos \theta) \sin^2 \theta \, d\theta \log D_p(0) + O\left(\frac{E_p(\sigma)}{p^2}\right). \end{aligned}$$

From this and (4.4), we deduce

$$\frac{E_p'(\sigma)}{E_p(\sigma)} = \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{1}{2} g'\left(\frac{\sigma}{p}\right) \log D_p(0) + O\left(\frac{1}{p^2}\right),$$

which implies the second estimate of (4.12). This completes the proof. \square

Lemma 4.3. *Let $J \geq 1$ be a fixed integer. Then we have*

$$\phi_1(\sigma, y) = 2 \log_2 \sigma + 2\gamma + \sum_{j=1}^J \frac{b_{j,1}}{(\log \sigma)^j} + O_J(R_J(\sigma, y))$$

uniformly for $y \geq \sigma \geq 3$, where the constant $b_{j,1}$ is given by

$$(4.15) \quad b_{j,1} := \int_0^\infty \frac{h'(u)}{u} (\log u)^{j-1} \, du$$

and $R_J(\sigma, y)$ is defined as in (1.20).

Proof. We have

$$\phi_1(\sigma, y) = \sum_{p \leq y} E_p'(\sigma) / E_p(\sigma).$$

Using the first relation of (4.12) for $p \leq \sigma^{2/3}$ and the second for $\sigma^{2/3} < p \leq y$, we obtain

$$\phi_1(\sigma, y) = \sum_{p \leq \sigma^{2/3}} \log D_p(0) + \frac{1}{2} \sum_{\sigma^{2/3} < p \leq y} g'\left(\frac{\sigma}{p}\right) \log D_p(0) + O\left(\frac{1}{\sigma^{1/3}}\right).$$

In view of (3.7), the preceding formula can be written as

$$(4.16) \quad \phi_1(\sigma, y) = \sum_{p \leq \sigma} \log D_p(0) + \sum_{\sigma^{2/3} < p \leq y} h'\left(\frac{\sigma}{p}\right) \log \left(1 - \frac{1}{p}\right)^{-1} + O\left(\frac{1}{\sigma^{1/3}}\right).$$

Similarly to (4.10), we can prove that

$$(4.17) \quad \sum_{\sigma^{2/3} < p \leq y} h'\left(\frac{\sigma}{p}\right) \log \left(1 - \frac{1}{p}\right)^{-1} = \sum_{j=1}^J \frac{b_{j,1}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)),$$

using (3.10), (3.11) and (4.11) instead of (3.9), (3.10) and (4.8). Now the desired result follows from (4.16), (4.10) and (4.17). \square

Lemma 4.4. *We have*

$$(4.18) \quad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \begin{cases} O\left(\frac{1}{\sigma^2}\right) & \text{if } p \leq \sigma^{1/2}, \\ \frac{1}{p^2}g''\left(\frac{\sigma}{p}\right) + O\left(\min\left\{\frac{1}{\sigma^2 p}, \frac{1}{\sigma p^2}\right\}\right) & \text{if } p > \sigma^{1/2}, \end{cases}$$

where $g(u)$ is defined as in (3.5).

Proof. First we write

$$(4.19) \quad \begin{aligned} E_p''(\sigma) &= \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta) \sin^2 \theta \, d\theta \\ &= E_p(\sigma) \log^2 D_p(0) + R'', \end{aligned}$$

where

$$R'' := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \left(\log^2 D_p(\theta) - \log^2 D_p(0) \right) \sin^2 \theta \, d\theta.$$

Using (4.13) and (4.19), we can deduce

$$(4.20) \quad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{R'' - 2R' \log D_p(0)}{E_p(\sigma)} - \left(\frac{R'}{E_p(\sigma)} \right)^2,$$

where R' is defined as in (4.14).

From the definitions of R' and R'' , a simple calculation shows that

$$R'' - 2R' \log D_p(0) = \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log^2 \left(\frac{D_p(\theta)}{D_p(0)} \right) \sin^2 \theta \, d\theta.$$

Since

$$\log^2 \left(\frac{D_p(\theta)}{D_p(0)} \right) = \log^2 \left(1 + \frac{2p(1 - \cos \theta)}{(p-1)^2} \right) = \frac{4(1 - \cos \theta)^2}{p^2} + O\left(\frac{(1 - \cos \theta)^2}{p^3} \right),$$

we have

$$R'' - 2R' \log D_p(0) = \frac{4}{p^2} E_{p,2}(\sigma) + O\left(\frac{E_{p,2}(\sigma)}{p^3} \right),$$

where $E_{p,j}(\sigma)$ is defined as in (3.2). By using (3.3) with the choice of $j = 2$ and the trivial estimate $E_{p,2}(\sigma) \leq 4E_p(\sigma)$, we deduce

$$(4.21) \quad \frac{R'' - 2R' \log D_p(0)}{E_p(\sigma)} = \frac{4}{p^2} \frac{E_{p,2}(\sigma)}{E_p(\sigma)} + O\left(\min\left\{ \frac{1}{\sigma^2 p}, \frac{1}{p^3} \right\} \right).$$

Similarly we have

$$\log \left(\frac{D_p(\theta)}{D_p(0)} \right) = -\log \left(1 + \frac{2p(1 - \cos \theta)}{(p-1)^2} \right) = -\frac{2(1 - \cos \theta)}{p} + O\left(\frac{(1 - \cos \theta)}{p^2} \right),$$

and therefore

$$R' = -\frac{2}{p} E_{p,1}(\sigma) + O\left(\frac{E_{p,1}(\sigma)}{p^2} \right).$$

Now (3.3) with $j = 1$ and the trivial estimate $E_{p,1}(\sigma) \leq 2E_p(\sigma)$ imply

$$(4.22) \quad \begin{aligned} \left(\frac{R'}{E_p(\sigma)} \right)^2 &= \frac{4}{p^2} \left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)} \right)^2 + O\left(\frac{E_{p,1}(\sigma)^2}{p^3 E_p(\sigma)^2} \right) \\ &= \frac{4}{p^2} \left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)} \right)^2 + O\left(\min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{p^3} \right\} \right). \end{aligned}$$

Inserting (4.21) and (4.22) into (4.20) and in view of (4.14), we deduce

$$(4.23) \quad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{4}{p^2} h_p(\sigma) + O\left(\min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{p^3} \right\} \right)$$

for all p and $\sigma > 0$, where

$$h_p(\sigma) := \frac{E_{p,2}(\sigma)}{E_p(\sigma)} - \left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)} \right)^2.$$

When $p \leq \sigma^{1/2}$, the inequality (3.3) of Lemma 3.2 implies that $h_p(\sigma) \ll (p/\sigma)^2$. From this and (4.23) we deduce the first estimate of (4.18).

If $p \geq \sigma^{1/2}$, we can use (4.3), (3.11) and (3.8) to write

$$\begin{aligned} 4h_p(\sigma) &= g''\left(\frac{\sigma}{p}\right) \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \\ &= g''\left(\frac{\sigma}{p}\right) + O\left(\min \left\{ \frac{\sigma}{p^2}, \frac{1}{\sigma} \right\} \right). \end{aligned}$$

Inserting it into (4.23) and in view of Lemma 3.1, we get, for $p \geq \sigma^{1/2}$,

$$\frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{1}{p^2} g''\left(\frac{\sigma}{p}\right) + O\left(\min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{\sigma p^2} \right\} \right).$$

This completes the proof. □

Lemma 4.5. *Let $J \geq 1$ be a fixed integer. Then we have*

$$\phi_2(\sigma, y) = \frac{1}{\sigma} \left\{ \sum_{j=1}^J \frac{b_{j,2}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}$$

uniformly for $y \geq \sigma \geq 2$, where

$$b_{j,2} := \int_0^\infty h''(u)(\log u)^{j-1} du.$$

In particular $b_{1,2} = 2$.

Proof. From Lemma 4.4 and (3.8), we deduce easily that

$$\begin{aligned} \phi_2(\sigma, y) &= \sum_{p \leq y} \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} \\ &= \sum_{\sigma^{1/2} < p \leq y} \frac{g''(\sigma/p)}{p^2} + O\left(\frac{1}{\sigma^{3/2} \log \sigma} \right) \\ &= \sum_{\sigma^{1/2} < p \leq y} \frac{h''(\sigma/p)}{p^2} + O\left(\frac{1}{\sigma^{3/2} \log \sigma} \right). \end{aligned}$$

Similarly to (4.10), we can prove that

$$\sum_{\sigma^{1/2} < p \leq y} \frac{h''(\sigma/p)}{p^2} = \frac{1}{\sigma} \left\{ \sum_{j=1}^J \frac{b_{j,2}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\},$$

by using (3.11), (3.12) and (4.8). Now the desired result follows from the preceding two estimates.

Finally

$$\begin{aligned} b_{1,2} &= \int_0^1 h''(u) du + \int_1^\infty h''(u) du \\ &= h'(1-) - h'(1+) = h'(1-) - (h'(1-) - 2) = 2. \end{aligned}$$

This completes the proof. \square

Similarly (even more easily, since we only need an upper bound instead of an asymptotic formula), we can prove the following result.

Lemma 4.6. *We have*

$$(4.24) \quad \phi_n(\sigma, y) \ll 1/(\sigma^{n-1} \log \sigma) \quad (n = 3, 4)$$

uniformly for $y \geq \sigma \geq 3$.

§ 5. Estimate of $|E(\kappa + i\tau, y)|$

Lemma 5.1. *For any $\delta \in (0, \frac{1}{4})$, there are two absolute positive constants c_1, c_2 and a positive constant $c_3 = c_3(\delta)$ such that for all $y \geq \sigma \geq 3$ we have*

$$(5.1) \quad \left| \frac{E(\sigma + i\tau, y)}{E(\sigma, y)} \right| \leq \begin{cases} 1 & \text{if } |\tau| \leq c_1 \sigma^{1/2} \log \sigma \text{ or } |\tau| \geq y^{1/\delta}, \\ e^{-c_2 \tau^2 / [\sigma (\log \sigma)^2]} & \text{if } c_1 \sigma^{1/2} \log \sigma \leq |\tau| \leq \sigma, \\ e^{-c_3 |\tau|^\delta} & \text{if } \sigma \leq |\tau| \leq y^{1/\delta}. \end{cases}$$

Proof. First we write

$$\begin{aligned} E_p(s) &= \frac{2}{\pi} \int_0^\pi (D_p(\theta)^{-1})^{-s} \sin^2 \theta d\theta \\ &= \frac{2}{\pi} \int_0^\pi \frac{\sin^2 \theta}{(1-s)(D_p(\theta)^{-1})'} d(D_p(\theta)^{-1})^{1-s}. \end{aligned}$$

Since $(D_p(\theta)^{-1})' = 2p^{-1} \sin \theta$, after a simplification and an integration by parts it follows that

$$\begin{aligned} E_p(s) &= \frac{p}{\pi(s-1)} \int_0^\pi D_p(\theta)^{s-1} \cos \theta d\theta \\ &= \frac{p}{\pi(s-1)} \int_0^{\pi/2} \{D_p(\theta)^{s-1} - D_p(\pi-\theta)^{s-1}\} \cos \theta d\theta. \end{aligned}$$

This implies that

$$(5.2) \quad \left| \frac{E_p(s)}{E_p(\sigma)} \right| = \left| \frac{\sigma-1}{s-1} \right| \left| \frac{E_p^*(s)}{E_p^*(\sigma)} \right|$$

with

$$E_p^*(s) := \int_0^{\pi/2} \{D_p(\theta)^{s-1} - D_p(\pi - \theta)^{s-1}\} \cos \theta \, d\theta.$$

1° Case of $\sigma^{1/\delta} < |\tau| \leq y^{1/\delta}$

Write

$$E_p^*(s) = \int_0^{\pi/2} D_p(\theta)^{s-1} \{1 - \Delta_p(\theta)^{s-1}\} \cos \theta \, d\theta$$

with

$$\Delta_p(\theta) := \frac{1 - 2p^{-1} \cos \theta + p^{-2}}{1 + 2p^{-1} \cos \theta + p^{-2}}.$$

It is clear that for all p , the function $\theta \mapsto \Delta_p(\theta)$ is increasing on $[0, \pi/2]$. It follows that

$$\begin{aligned} E_p^*(\sigma) &\geq \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \{1 - \Delta_p(\theta)^{\sigma-1}\} \cos \theta \, d\theta \\ &\geq \{1 - \Delta_p(\pi/4)^{\sigma-1}\} \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \end{aligned}$$

for all p and $\sigma \geq 1$. This implies that

$$(5.3) \quad \left| \frac{1}{E_p^*(\sigma)} \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \right| \leq \frac{1}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

Similarly since the function $\theta \mapsto D_p(\theta)^{\sigma-1} \cos \theta$ is decreasing on $[0, \pi/2]$ for all p and $\sigma \geq 2$, we can deduce, via (5.3), that

$$(5.4) \quad \left| \frac{1}{E_p^*(\sigma)} \int_{\pi/4}^{\pi/2} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \right| \leq \frac{1}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

From (5.3) and (5.4), we deduce that

$$\left| \frac{E_p^*(s)}{E_p^*(\sigma)} \right| \leq \frac{2}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

It is easy to verify that for all $p \geq \sigma \geq 2$, we have

$$\Delta_p\left(\frac{\pi}{4}\right)^{\sigma-1} \leq \left(1 - \frac{\sqrt{2}}{p} + \frac{1}{p^2}\right)^{\sigma-1} \leq 1 - \frac{\sigma-1}{4p}.$$

Combining these estimates with (5.2), we obtain

$$\left| \frac{E_p(s)}{E_p(\sigma)} \right| \leq \frac{8p}{|s-1|} \leq \frac{p^4}{|\tau|} \quad (p \geq \sigma).$$

By multiplying this inequality for $\sigma < p \leq |\tau|^\delta$ ($\leq y$) and the trivial inequality $|E_p(s)| \leq |E_p(\sigma)|$ for the others p , we deduce, via the prime number theorem, that

$$\begin{aligned} \left| \frac{E(s, y)}{E(\sigma, y)} \right| &\leq \exp \left\{ - \sum_{\sigma < p \leq |\tau|^\delta} \log |\tau| + 4 \sum_{\sigma < p \leq |\tau|^\delta} \log p \right\} \\ &\leq e^{-\{1/\delta - 4 + o(1)\}|\tau|^\delta}. \end{aligned}$$

2° Case of $c_1\sigma^{1/2}\log\sigma \leq |\tau| \leq \sigma^{1/\delta}$

For $p \geq \sigma^{1/2} \geq 2$, we can write

$$\begin{aligned} |E_p^*(s)| &\leq \int_0^{\pi/2} \{D_p(\theta)^{\sigma-1} + D_p(\pi-\theta)^{\sigma-1}\} \cos\theta \, d\theta \\ &= \left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\} \int_0^{\pi/2} (e^{2[(\sigma-1)/p]\cos\theta} + e^{-2[(\sigma-1)/p]\cos\theta}) \cos\theta \, d\theta \end{aligned}$$

and

$$\begin{aligned} |E_p^*(\sigma)| &= \int_0^{\pi/2} \{D_p(\theta)^{\sigma-1} - D_p(\pi-\theta)^{\sigma-1}\} \cos\theta \, d\theta \\ &= \left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\} \int_0^{\pi/2} (e^{2[(\sigma-1)/p]\cos\theta} - e^{-2[(\sigma-1)/p]\cos\theta}) \cos\theta \, d\theta. \end{aligned}$$

From these, we deduce that

$$(5.5) \quad \left| \frac{E_p^*(s)}{E_p^*(\sigma)} \right| \leq \left\{1 + O\left(\frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}}\right)\right\} \quad (2 \leq \sigma^{1/2} \leq p \leq \sigma)$$

where we have used the following facts

$$\int_0^{\pi/2} e^{2[(\sigma-1)/p]\cos\theta} \cos\theta \, d\theta \gg e^{\sigma/p} \quad \text{and} \quad \int_0^{\pi/2} e^{-2[(\sigma-1)/p]\cos\theta} \cos\theta \, d\theta \ll 1.$$

Inserting (5.5) into (5.2), for $2 \leq \sigma^{1/2} \leq p \leq \sigma$ we obtain

$$\begin{aligned} \left| \frac{E_p(s)}{E_p(\sigma)} \right| &\leq \exp \left\{ -\log \left| \frac{s-1}{\sigma-1} \right| + C \left(\frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}} \right) \right\} \\ &\leq \begin{cases} e^{-\tau^2/(2\sigma^2) + C\sigma/p^2 + Ce^{-\sigma/p}} & \text{if } 3 \leq |\tau| \leq \sigma, \\ e^{-\frac{1}{2}\log(1+\tau^2/\sigma^2) + C\sigma/p^2 + Ce^{-\sigma/p}} & \text{if } \sigma \leq |\tau| \leq \sigma^{1/\delta}, \end{cases} \end{aligned}$$

where $C > 0$ is an absolute constant.

Now by multiplying these inequalities for $\sigma/(4\log\sigma) \leq p \leq \sigma/(2\log\sigma)$ and the trivial inequality $|E_p(s)| \leq E_p(\sigma)$ for the other p , we get

$$\begin{aligned} \left| \frac{E(s, y)}{E(\sigma, y)} \right| &\leq \exp \left\{ - \sum_{\sigma/(4\log\sigma) \leq p \leq \sigma/(2\log\sigma)} \left(\frac{\tau^2}{2\sigma^2} - \frac{C\sigma}{p^2} - \frac{C}{e^{\sigma/p}} \right) \right\} \\ &\leq \exp \left\{ - \left(\frac{\tau^2}{16\sigma(\log\sigma)^2} - 10C - \frac{10C}{\sigma \log\sigma} \right) \right\} \\ &\leq \exp \left\{ - \frac{c_2\tau^2}{\sigma(\log\sigma)^2} \right\} \end{aligned}$$

if $c_1\sigma^{1/2}\log\sigma \leq |\tau| \leq \sigma$, and

$$\begin{aligned} (5.6) \quad \left| \frac{E(s, y)}{E(\sigma, y)} \right| &\leq \exp \left\{ - \sum_{\sigma/(4\log\sigma) \leq p \leq \sigma/(2\log\sigma)} \left[\frac{1}{2} \log \left(1 + \frac{\tau^2}{\sigma^2} \right) - \frac{C\sigma}{p^2} - \frac{C}{e^{\sigma/p}} \right] \right\} \\ &\leq \exp \left\{ - \left[\frac{\sigma}{8\log\sigma} \log \left(1 + \frac{\tau^2}{\sigma^2} \right) - 10C - \frac{10C}{\sigma \log\sigma} \right] \right\} \\ &\leq \exp \{ -c_3|\tau|^\delta \} \end{aligned}$$

if $\sigma \leq |\tau| \leq \sigma^{1/\delta}$. This completes the proof. \square

§ 6. Proof of Theorem 3

We follow the argument of Granville & Soundararajan [4] to prove Theorem 3. We shall divide the proof in several steps which are embodied in the following lemmas.

The first one is a classic integration formula (see [4], page 1019).

Lemma 6.1. *Let $c > 0$, $\lambda > 0$ and $N \in \mathbb{N}$. Then we have*

$$(6.1) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < y < e^{-\lambda N}, \\ \in [0, 1] & \text{if } e^{-\lambda N} \leq y < 1, \\ 1 & \text{if } y \geq 1. \end{cases}$$

The second one is an analogue for (3.6) and (3.7) of [4] (see also Lemma 3.1 of [20]).

Lemma 6.2. *Let $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda \leq e^{-t}$. Then we have*

$$(6.2) \quad \Phi(t, y) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s}} \frac{ds}{s} \leq \Phi(te^{-\lambda}, y),$$

$$(6.3) \quad \Phi(te^{-\lambda}, y) - \Phi(t, y) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s}} \frac{e^{2\lambda s} - e^{-2\lambda s}}{\lambda s} \frac{ds}{s}.$$

Proof. Denote by $\mathbf{1}_X(\omega)$ the characteristic function of the set $X \subset \Omega$. Then by Lemma 6.1 with $N = 1$ and $c = \kappa$, we have

$$\mathbf{1}_{\{\omega \in \Omega: L(1, g^{\natural}(\omega); y) > (e^{\gamma t})^2\}}(\omega) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{L(1, g^{\natural}(\omega); y)}{(e^{\gamma t})^2} \right)^s \frac{e^{\lambda s} - 1}{\lambda s^2} ds.$$

Integrating over Ω and interchanging the order of integrations yield

$$\begin{aligned} \Phi(t, y) &\leq \int_{\Omega} \left\{ \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{L(1, g^{\natural}(\omega); y)}{(e^{\gamma t})^2} \right)^s \frac{e^{\lambda s} - 1}{\lambda s^2} ds \right\} d\mu(\omega) \\ &= \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s}} \frac{ds}{\lambda s^2}. \end{aligned}$$

This proves the first inequality of (6.2). The second can be treated by noticing that

$$\begin{aligned} \mathbf{1}_{\{\omega \in \Omega: L(1, g^{\natural}(\omega); y) > (e^{\gamma - \lambda} t)^2\}}(\omega) &= \mathbf{1}_{\{\omega \in \Omega: L(1, g^{\natural}(\omega); y) > (e^{\gamma t})^2\}}(\omega) \\ &\quad + \mathbf{1}_{\{\omega \in \Omega: (e^{\gamma t})^2 \geq L(1, g^{\natural}(\omega); y) > (e^{\gamma - \lambda} t)^2\}}(\omega) \\ &\geq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{L(1, g^{\natural}(\omega); y)}{(e^{\gamma t})^2} \right)^s \frac{e^{\lambda s} - 1}{\lambda s^2} ds. \end{aligned}$$

From (6.2), we can deduce

$$\begin{aligned} \Phi(te^{-\lambda}, y) - \Phi(t, y) &\leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma - \lambda} t)^{2s}} \frac{ds}{\lambda s^2} - \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma + \lambda} t)^{2s}} \frac{ds}{\lambda s^2} \\ &= \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s}} \frac{e^{2\lambda s} - e^{-2\lambda s}}{\lambda s^2} ds. \end{aligned}$$

This completes the proof. \square

Lemma 6.3. *Let $t \geq 1$, $y \geq 2e^t$ and $0 < \kappa\lambda \leq 1$. Then we have*

$$\frac{1}{2\pi i} \int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{E(s, y)}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} ds = \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2} (e^{\gamma t})^{2\kappa}} \left\{ 1 + O\left(\kappa\lambda + \frac{\log \kappa}{\kappa}\right) \right\}.$$

Proof. First in view of (4.24) we write, for $s = \kappa + i\tau$ and $|\tau| \leq \kappa$,

$$E(s, y) = \exp \left\{ \sigma_0 + i\sigma_1\tau - \frac{\sigma_2}{2}\tau^2 - i\frac{\sigma_3}{6}\tau^3 + O(\sigma_4\tau^4) \right\}$$

and

$$\frac{e^{\lambda s} - 1}{\lambda s^2} = \frac{1}{\kappa} \left\{ 1 - \frac{i}{\kappa}\tau + O\left(\kappa\lambda + \frac{\tau^2}{\kappa^2}\right) \right\}.$$

Since $\sigma_1 = \log t + \gamma$, we have

$$\frac{E(s, y)}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} = \frac{E(\kappa, y)}{\kappa (e^{\gamma t})^{2\kappa}} e^{-(\sigma_2/2)\tau^2} \left\{ 1 - \frac{i}{\kappa}\tau - i\frac{\sigma_3}{6}\tau^3 + O(R(\tau)) \right\}$$

with

$$R(\tau) := \kappa\lambda + \kappa^{-2}\tau^2 + \sigma_4\tau^4 + \sigma_3^2\tau^6.$$

Now we integrate the last expression over $|\tau| \leq \kappa$ to obtain

$$(6.4) \quad \frac{1}{2\pi i} \int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{E(s, y)}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} ds = \frac{E(\kappa, y)}{2\pi\kappa (e^{\gamma t})^{2\kappa}} \int_{-\kappa}^{\kappa} e^{-(\sigma_2/2)\tau^2} \{1 + O(R(\tau))\} d\tau,$$

where we have used the fact that the integrals involving $(i/\kappa)\tau$ and $(i\sigma_3/6)\tau^3$ vanish.

On the other hand, using lemmas 4.5 and 4.6 we have

$$\begin{aligned} \int_{-\kappa}^{\kappa} e^{-(\sigma_2/2)\tau^2} d\tau &= \sqrt{\frac{2\pi}{\sigma_2}} \left\{ 1 + O\left(\exp\left\{-\frac{1}{2}\kappa^2\sigma_2\right\}\right) \right\}, \\ \int_{\kappa-i\kappa}^{\kappa+i\kappa} e^{-(\sigma_2/2)\tau^2} R(\tau) d\tau &\ll \frac{1}{\sqrt{\sigma_2}} \left(\kappa\lambda + \frac{1}{\kappa^2\sigma_2} + \frac{\sigma_3^2}{\sigma_2^3} + \frac{\sigma_4}{\sigma_2^2} \right) \\ &\ll \frac{1}{\sqrt{\sigma_2}} \left(\kappa\lambda + \frac{\log \kappa}{\kappa} \right). \end{aligned}$$

Inserting these into (6.4), we obtain the desired result. □

Lemma 6.4. *Let δ and c_3 be two constants determined by Lemma 5.1. Then we have*

$$(6.5) \quad \int_{\kappa \pm i\kappa}^{\kappa \pm i\infty} \frac{E(s, y)}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} ds \ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} R_1,$$

$$(6.6) \quad \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y)}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} (e^{2\lambda s} - e^{-2\lambda s}) ds \ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} R_2,$$

uniformly for $t \geq 1$, $y \geq 2e^t$, $\kappa \geq 2$ and $0 < \lambda\kappa \leq 1$, where

$$\begin{aligned} R_1 &:= \lambda^{-1} e^{-c_3\kappa^\delta} + \lambda^{-1} (\kappa/\log \kappa)^{1/2} y^{-1/\delta}, \\ R_2 &:= \lambda\kappa (\log \kappa)^{1/2} + e^{-(c_3/2)\kappa^\delta} + \lambda^{-1} (\kappa/\log \kappa)^{1/2} y^{-1/\delta}. \end{aligned}$$

Proof. We split the integral in (6.5) into two parts according to $\kappa \leq |\tau| \leq y^{1/\delta}$ or $|\tau| \geq y^{1/\delta}$. Using Lemma 5.1 with $\sigma = \kappa$ and the inequality $(e^{\lambda s} - 1)/s^2 \ll 1/\tau^2$, the integral in (6.5) is

$$\ll \frac{E(\kappa, y)}{(e^{\gamma t})^{2\kappa} \lambda} \left(\frac{e^{-c_3 \kappa^\delta}}{\kappa} + \frac{1}{y^{1/\delta}} \right),$$

which implies (6.5), in view of Lemma 4.5 with $J = 1$.

Similarly we split the integral in (6.6) into four parts according to

$$|\tau| \leq c_1 \kappa^{1/2} \log \kappa, \quad c_1 \kappa^{1/2} \log \kappa < |\tau| \leq \kappa, \quad \kappa < |\tau| \leq y^{1/\delta}, \quad |\tau| \geq y^{1/\delta}.$$

By Lemma 5.1 with $\sigma = \kappa$ and the inequalities

$$(e^{\lambda s} - 1)/\lambda s \ll \min\{1, 1/(\lambda|\tau|)\},$$

$$(e^{2\lambda s} - e^{-2\lambda s})/s \ll \min\{\lambda, 1/|\tau|\},$$

the integral in (6.6) is, as before,

$$\ll_\varepsilon \frac{E(\kappa, y)}{(e^{\gamma t})^{2\kappa}} \left(\lambda \kappa^{1/2} \log \kappa + e^{-c_3 \kappa^\delta} + \lambda^{-1} y^{-1/\delta} \right),$$

which implies (6.6), as before. \square

Now we are ready to complete the proof of Theorem 3. Lemma 6.3 and (6.5) of Lemma 6.4 give

$$(6.7) \quad \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y)}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} ds = \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2}(e^{\gamma t})^{2\kappa}} \{1 + O(R')\}$$

where

$$R' := \frac{\log \kappa}{\kappa} + \kappa \lambda + \frac{e^{-c_3 \kappa^\delta} + (\kappa/\log \kappa)^{1/2} y^{-1/\delta}}{\lambda}.$$

Taking $\lambda = \kappa^{-2}$ and noticing $y \geq 2e^t \asymp \kappa$ and $1/\delta > 4$, we deduce

$$(6.8) \quad R' \ll t/e^t.$$

Combining (6.7) and (6.8) with (6.2), we obtain

$$(6.9) \quad \Phi(t, y) \leq \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2}(e^{\gamma t})^{2\kappa}} \left\{ 1 + O\left(\frac{t}{e^t}\right) \right\} \leq \Phi(te^{-\lambda}, y)$$

uniformly for $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda \leq e^{-t}$.

On the other hand, (6.3) of Lemma 6.2 and (6.6) of Lemma 6.4 imply

$$\begin{aligned} \Phi(te^{-\lambda}, y) - \Phi(t, y) &\ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2}(e^{\gamma t})^{2\kappa}} \left(\lambda \kappa (\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3 \kappa^\delta}} + \frac{(\kappa/\log \kappa)^{1/2}}{\lambda y^{1/\delta}} \right) \\ &\ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2}(e^{\gamma t})^{2\kappa}} \left(\lambda \kappa (\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3 \kappa^\delta}} \right) \end{aligned}$$

when $y^{-1/(2\delta)} \kappa^{-1/2} (\log \kappa)^{-1} \leq \lambda \leq \kappa^{-1}$. Since $\Phi(te^{-\lambda}, y) - \Phi(t, y)$ is a non-decreasing function of λ , we deduce

$$(6.10) \quad \Phi(te^{-\lambda}, y) - \Phi(t, y) \ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2}(e^{\gamma t})^{2\kappa}} \left(\lambda \kappa (\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3 \kappa^\delta}} + \frac{\kappa (\log \kappa)^{1/2}}{y^{1/(2\delta)}} \right)$$

uniformly for $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda \leq e^{-t}$. Obviously the estimates (6.9) and (6.10) imply the desired result. This completes the proof of Theorem 3. \square

§ 7. Proof of Theorem 4

Using Lemmas 4.1 and 4.5, we can write

$$\begin{aligned} \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}} &= \exp \left\{ \phi(\kappa, y) - 2\kappa(\gamma + \log t) + O(\log \kappa) \right\} \\ &= \exp \left\{ \kappa \left(2 \log_2 \kappa - 2 \log t + \sum_{j=1}^J \frac{b_{j,0}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right) \right\}. \end{aligned}$$

On the other hand, Lemma 4.3 and (1.17) imply that

$$2 \log_2 \kappa + 2\gamma + \sum_{j=1}^J \frac{b_{j,1}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) = 2(\log t + \gamma).$$

Combining these estimates, we can obtain

$$\frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2}(e^\gamma t)^{2\kappa}} = \exp \left\{ -\kappa \left[\sum_{j=1}^J \frac{b_{j,1} - b_{j,0}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right] \right\}.$$

In view of (1.21), (4.2) and (4.15), we have $b_{j,1} - b_{j,0} = a_j$. This completes the proof. \square

§ 8. Proof of Corollary 5

We first prove an asymptotic developpment of $\kappa(t, y)$ in t .

Lemma 8.1. *For each integer $J \geq 1$, there are computable constants $\gamma_0, \gamma_1, \dots, \gamma_J$ such that the asymptotic formula*

$$(8.1) \quad \kappa(t, y) = e^{t-\gamma_0} \left\{ 1 + \sum_{j=1}^J \frac{\gamma_j}{t^j} + O_J(R_J^*(t, y)) \right\}$$

holds uniformly for $t \geq 1$ and $y \geq 2e^t$, where

$$R_N^*(t, y) := \frac{1}{t^{N+1}} + \frac{e^t t}{y \log y}.$$

Further γ_0 is given by (1.24) and $\gamma_1 = -\frac{1}{8}b_{1,1}^2 - \frac{1}{4}b_{2,1}$.

Proof. By Lemma 4.3 and (1.17), we have

$$(8.2) \quad 2 \log t = 2 \log_2 \kappa + \sum_{j=1}^{J+1} \frac{b_{j,1}}{(\log \kappa)^j} + O_J(R_{J+1}(\kappa, y)),$$

where $R_J(\kappa, y)$ is defined as in (1.20). From (8.2), we easily deduce that

$$\begin{aligned} t &= (\log \kappa) \prod_{j=1}^{J+1} \exp \left\{ \frac{b_{j,1}}{2(\log \kappa)^j} \right\} \exp \{ O_J(R_{J+1}(\kappa, y)) \} \\ &= (\log \kappa) \prod_{j=1}^{J+1} \left\{ \sum_{m_j=0}^{J+1} \frac{1}{m_j!} \left(\frac{b_{j,1}}{2(\log \kappa)^j} \right)^{m_j} + O_J(R_{J+1}(\kappa, y)) \right\}. \end{aligned}$$

Developping the product, we get

$$t = (\log \kappa) \left\{ \sum_{j=0}^{J+1} \frac{b'_j}{(\log \kappa)^j} + O_J(R_{J+1}(\kappa, y)) \right\},$$

where

$$\begin{aligned} b'_j &:= \sum_{\substack{m_1 \geq 0, \dots, m_{J+1} \geq 0 \\ m_1 + 2m_2 + \dots + (J+1)m_{J+1} = j}} \frac{b_{1,1}^{m_1} \dots b_{J+1,1}^{m_{J+1}}}{(2m_1)!! \dots (2m_{J+1})!!} \\ &= \sum_{\substack{m_1 \geq 0, \dots, m_j \geq 0 \\ m_1 + 2m_2 + \dots + jm_j = j}} \frac{b_{1,1}^{m_1} \dots b_{j,1}^{m_j}}{(2m_1)!! \dots (2m_j)!!}. \end{aligned}$$

Since $b'_0 = 1$ and $b'_1 = b_{1,1}/2 = \gamma_0$, the preceeding asymptotic formula can be written as

$$(8.3) \quad t = \log \kappa + \gamma_0 + \sum_{j=1}^J \frac{b'_{j+1}}{(\log \kappa)^j} + O_J(R_J^*(t, y)),$$

where we have used the fact that $\kappa(t, y) \asymp e^t$ (see Lemma 2.3) and $(\log k)R_{J+1}(\kappa, y) \asymp R_J^*(t, y)$.

With the help of (8.3), a simple recurrence argument shows that there are constants γ'_n such that

$$(8.4) \quad t = \log \kappa + \sum_{j=0}^J \frac{\gamma'_j}{t^j} + O_J(R_J^*(t, y)).$$

In fact taking $J = 0$ in (8.3), we see that (8.4) holds for $J = 0$. Suppose that it holds for $0, \dots, J-1$, i.e.

$$t = \log \kappa + \sum_{i=0}^{J-j-1} \frac{\gamma'_i}{t^i} + O(R_{J-j-1}^*(t, y)) \quad (j = 0, \dots, J-1),$$

which is equivalent to

$$(8.5) \quad \log \kappa = t \left\{ 1 - \sum_{i=1}^{J-j} \frac{\gamma'_{i-1}}{t^i} + O\left(\frac{R_{J-j-1}^*(t, y)}{t} \right) \right\} \quad (j = 0, \dots, J-1).$$

This holds also for $j = J$ if we use the convention:

$$\sum_{i=0}^{-1} = 0 \quad \text{and} \quad R_{-1}^*(t, y) := 1,$$

since $\log \kappa = t + O(1)$. Inserting it into (8.3), we easily see that (8.4) holds also for J . In particular we have

$$\gamma'_1 = b'_2 = \frac{1}{8}b_{1,1}^2 + \frac{1}{4}b_{2,1}.$$

Now (8.1) is an immediate consequence of (8.4) with

$$\gamma_j := \sum_{\substack{m_1 \geq 0, \dots, m_J \geq 0 \\ m_1 + 2m_2 + \dots + Jm_J = j}} (-1)^{m_1 + \dots + m_J} \frac{\gamma_1'^{m_1} \dots \gamma_J'^{m_J}}{m_1! \dots m_J!}.$$

This completes the proof. \square

Now we are ready to prove Corollary 5.

Using (8.5), we have

$$(8.6) \quad \begin{aligned} \sum_{j=1}^J \frac{a_j}{(\log \kappa)^j} &= \sum_{j=1}^J \frac{a_j}{t^j} \left\{ 1 - \sum_{i=1}^{J-j} \frac{\gamma_{i-1}'}{t^i} + O_N \left(\frac{R_{J-j-1}^*(t, y)}{t} \right) \right\}^{-j} \\ &= \sum_{j=1}^J \frac{\rho_j}{t^j} + O_J \left(\frac{R_{J-2}^*(t, y)}{t^2} \right), \end{aligned}$$

where the ρ_n are constants. In particular we have $\rho_1 = a_1 = 1$ and $\rho_2 = \gamma_0 + a_2$.

Now Theorem 4, (8.1) and (8.6) imply the result of Corollary with

$$a_1^* = \rho_1 = 1, \quad a_j^* = \rho_j + \sum_{i=1}^{j-1} \gamma_i \rho_{j-i} \quad (j \geq 2).$$

This completes the proof of Corollary 5. \square

§ 9. Proof of Theorem 2

For each $\eta \in (0, \frac{1}{2})$, define

$$\mathbf{H}_k^+(1; \eta) := \{f \in \mathbf{H}_k^*(1) : L(s, f) \neq 0, s \in \mathcal{S}\},$$

where $\mathcal{S} := \{s := \sigma + i\tau : \sigma \geq 1 - \eta, |\tau| \leq 100k^\eta\} \cup \{s := \sigma + i\tau : \sigma \geq 1, \tau \in \mathbb{R}\}$, and

$$\mathbf{H}_k^-(1; \eta) := \mathbf{H}_k^*(1) \setminus \mathbf{H}_k^+(1; \eta).$$

Then we have (see [10], (1.11))

$$(9.1) \quad |\mathbf{H}_k^-(1; \eta)| \ll_\eta k^{31\eta}.$$

Our starting point in the proof of Theorem 2 is the evaluation of the moments of $L(1, f)$. For this, we recall a particular case of Proposition 6.1 of [10].

Lemma 9.1. *Let $\eta \in (0, \frac{1}{31})$ be fixed. There are two positive constants $c_i = c_i(\eta)$ ($i = 4, 5$) such that*

$$(9.2) \quad \sum_{f \in H_k^+(1; \eta)} \omega_f L(1, f)^s = E(s) + O_\eta(e^{-c_4 \log k / \log_2 k})$$

uniformly for

$$(9.3) \quad k \geq 16, \quad 2 \mid k \quad \text{and} \quad |s| \leq 2T_k$$

with

$$T_k := c_5 \log k / (\log_2 k \log_3 k).$$

Here $E(s)$ is defined by (1.15).

Let $\kappa(t, y)$ be the saddle-point determined by (1.17) and $\kappa_t := \kappa(t, \infty)$. For $k \geq 16, 2 \mid k$, $\lambda > 0$, $N \in \mathbb{N}$ and $t > 0$, introduce the two integrals

$$I_1(k, t; \lambda, N) := \frac{1}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \sum_{f \in H_k^+(1; \eta)} \omega_f \left(\frac{L(1, f)}{(e^\gamma t)^2} \right)^s \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s}$$

and

$$I_2(k, t; \lambda, N) := \frac{1}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \frac{E(s)}{(e^\gamma t)^{2s}} \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s}.$$

Lemma 9.2. *Let $\eta \in (0, \frac{1}{200}]$ be fixed. Then we have*

$$(9.4) \quad \tilde{F}_k(t) + O_\eta(k^{-5/6}) \leq I_1(k, t; \lambda, N) \leq \tilde{F}_k(te^{-\lambda N}) + O_\eta(k^{-5/6}),$$

$$(9.5) \quad \Phi(t) \leq I_2(k, t; \lambda, N) \leq \Phi(te^{-\lambda N})$$

uniformly for $k \geq 16, 2 \mid k$, $\lambda > 0$, $N \in \mathbb{N}$ and $t > 0$. The implied constants depend on η only.

Proof. By exchanging the order of summation and by using Lemma 6.1 with $c = \kappa_t$, we obtain

$$\begin{aligned} I_1(k, t; \lambda, N) &= \sum_{f \in H_k^+(1; \eta)} \frac{\omega_f}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \left(\frac{L(1, f)}{(e^\gamma t)^2} \right)^s \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s}, \\ &\geq \sum_{f \in H_k^+(1; \eta), L(1, f) \geq (e^\gamma t)^2} \omega_f. \end{aligned}$$

In view of the second estimate of (1.7) and of (9.1), we reintroduce the missing forms

$$\begin{aligned} I_1(k, t; \lambda, N) &\geq \sum_{f \in H_k^*(1), L(1, f) \geq (e^\gamma t)^2} \omega_f + O\left(\sum_{f \in H_k^* \setminus H_k^+(1; \eta)} \omega_f \right) \\ &\geq \sum_{f \in H_k^*(1), L(1, f) \geq (e^\gamma t)^2} \omega_f + O(k^{-1+31\eta} \log k). \end{aligned}$$

Clearly this implies the first inequality of (9.4), thanks to (1.6) and (1.7).

Similarly, using Lemma 6.1 with $c = \kappa_t$, we find

$$\begin{aligned} I_1(k, t; \lambda, N) &\leq \sum_{\substack{f \in H_k^+(1; \eta) \\ L(1, f) \geq (e^\gamma t)^2}} \omega_f + \sum_{\substack{f \in H_k^+(1; \eta) \\ (e^\gamma te^{-\lambda N})^2 \leq L(1, f) < (e^\gamma t)^2}} \omega_f \\ &= \sum_{\substack{f \in H_k^+(1; \eta) \\ L(1, f) \geq (e^\gamma te^{-\lambda N})^2}} \omega_f. \end{aligned}$$

As before, we can easily show that the last sum is $\leq \tilde{F}_k(te^{-\lambda N}) + O(k^{-5/6})$.

The estimates (9.5) can be proved in the same way as (6.2). \square

Lemma 9.3. *Let $\eta \in (0, \frac{1}{200}]$ be fixed and c_4 be the positive constant given by Lemma 9.1. Then we have*

$$(9.6) \quad |I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N)| \ll e^{-c_4(\log k)/\log_2 k} \frac{(1 + e^{\lambda\kappa_t})^{2N} \log T_k}{(e\gamma t)^{2\kappa_t}} + \frac{E(\kappa_t) + e^{-c_4(\log k)/\log_2 k} \left(\frac{1 + e^{\lambda\kappa_t}}{\lambda T_k} \right)^{2N}}{N(e\gamma t)^{2\kappa_t}}$$

uniformly for $\lambda > 0$, $N \in \mathbb{N}$, $k \geq 16$, $2 \mid k$ and $t \leq T(k)$, where $T(k)$ is given by (1.10). The implied constant depends on η only.

Proof. By the definitions of I_1 and I_2 , we can write

$$\begin{aligned} & I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N) \\ &= \frac{1}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \left(\sum_{f \in H_k^+(1; \eta)} \omega_f L(1, f)^s - E(s) \right) \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s(e\gamma t)^{2s}}. \end{aligned}$$

In order to estimate the last integral, we split it into two parts according to $|\tau| \leq T_k$ or $|\tau| > T_k$.

In view of (1.18), it is easy to see that $\kappa_t \leq T_k$ for $t \leq T(k)$. Thus we may apply (9.2) of Lemma 9.1 for $s = \kappa_t + i\tau$ with $|\tau| \leq T_k$. Note that $|(e^{\lambda s} - 1)/(\lambda s)| \leq 1 + e^{\lambda\kappa_t}$ for $s = \kappa_t + i\tau$, which is easily seen by looking at the cases $|\lambda s| \leq 1$ and $|\lambda s| > 1$. The contribution of $|\tau| \leq T_k$ to $|I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N)|$ is

$$(9.7) \quad \ll e^{-c_4(\log k)/\log_2 k} \frac{(1 + e^{\lambda\kappa_t})^{2N} \log T_k}{(e\gamma t)^{2\kappa_t}}.$$

Since $\kappa_t \leq T_k$ for $t \leq T(k)$, we can apply (9.2) of Lemma 9.1 to write, for $s = \kappa_t + i\tau$ with $\tau \in \mathbb{R}$,

$$\begin{aligned} \left| \sum_{f \in H_k^+(1; \eta)} \omega_f L(1, f)^s - E(s) \right| &\leq \sum_{f \in H_k^+(1; \eta)} \omega_f L(1, f)^{\kappa_t} + E(\kappa_t) \\ &\leq 2E(\kappa_t) + O(e^{-c_4(\log k)/\log_2 k}). \end{aligned}$$

Thus the contribution of $|\tau| > T_k$ to $|I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N)|$ is

$$(9.8) \quad \begin{aligned} &\ll \frac{E(\kappa_t) + e^{-c_4(\log k)/\log_2 k}}{(e\gamma t)^{2\kappa_t}} \int_{|\tau| \geq T_k} \left(\frac{1 + e^{\lambda\kappa_t}}{\lambda|\tau|} \right)^{2N} \frac{d\tau}{|\tau|} \\ &\ll \frac{E(\kappa_t) + e^{-c_4(\log k)/\log_2 k}}{N(e\gamma t)^{2\kappa_t}} \left(\frac{1 + e^{\lambda\kappa_t}}{\lambda T_k} \right)^{2N}. \end{aligned}$$

Combining (9.7) and (9.8) yields to the required estimate. \square

End of the proof of Theorem 2

For simplicity of notation, we write

$$I_j := I_j(k, t; \lambda, N) \quad \text{and} \quad I_j^+ := I_j(k, te^{\lambda N}; \lambda, N) \quad (j = 1, 2).$$

By using Lemma 9.2, we have

$$(9.9) \quad \begin{aligned} \tilde{F}_k(t) &\leq I_1 + O(k^{-5/6}) \\ &= I_2 + O(|I_1 - I_2| + k^{-5/6}) \\ &\leq \Phi(te^{-\lambda N}) + O(|I_1 - I_2| + k^{-5/6}) \\ &\leq \Phi(t) + |\Phi(te^{-\lambda N}) - \Phi(t)| + O(|I_1 - I_2| + k^{-5/6}) \end{aligned}$$

and

$$\begin{aligned}
 \widetilde{F}_k(t) &\geq I_1^+ + O(k^{-5/6}) \\
 &= I_2^+ + O(|I_1^+ - I_2^+| + k^{-5/6}) \\
 &\geq \Phi(te^{\lambda N}) + O(|I_1^+ - I_2^+| + k^{-5/6}) \\
 &\geq \Phi(t) - |\Phi(t) - \Phi(te^{\lambda N})| + O(|I_1^+ - I_2^+| + k^{-5/6}).
 \end{aligned}
 \tag{9.10}$$

In view of (6.10) and Theorem 3, we have

$$|\Phi(t) - \Phi(te^{-\lambda N})| \ll \Phi(t) \{ \lambda N \kappa_t (\log \kappa_t)^{1/2} + e^{-(c_3/2)\kappa_t^\delta} \}$$

for $\lambda N \leq e^{-t}$. Take

$$\lambda = e^{5A}/T_k \quad \text{and} \quad N = \lfloor \log_2 k \rfloor.$$

Since $T_k = e^{T(k) + \frac{3}{2} \log_3 k + 2C + \log c_5}$, it is easy to see that

$$\lambda N \leq e^{-T(k) - 2C} T(k)^{-1/2} \quad \text{and} \quad \kappa_t \asymp e^t.$$

Inserting these estimates into the preceding inequality, a simple calculation shows that

$$|\Phi(t) - \Phi(te^{-\lambda N})| \leq \Phi(t) \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \},$$

provided the constant C is suitably large, where $c_6 = c_6(\eta, \delta)$ is a positive constant.

Similarly by using (6.10) with $te^{\lambda N}$ in place of t , we have

$$|\Phi(t) - \Phi(te^{\lambda N})| \ll \Phi(te^{\lambda N}) \{ \lambda N \kappa_{te^{\lambda N}} (\log \kappa_{te^{\lambda N}})^{1/2} + e^{-(c_3/2)\kappa_{te^{\lambda N}}^\delta} \}.$$

Since for $t \leq T(k)$ we have

$$te^{\lambda N} = t + O((\log_2 k)^3 (\log_3 k) / \log k) \quad \text{and} \quad \kappa_{te^{\lambda N}} \asymp e^{te^{\lambda N}} \asymp e^t,$$

the preceding estimate can be written as

$$\begin{aligned}
 |\Phi(t) - \Phi(te^{\lambda N})| &\leq \frac{1}{4} \Phi(te^{\lambda N}) \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \} \\
 &\leq \frac{1}{4} \Phi(t) \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \} \\
 &\quad + \frac{1}{4} |\Phi(t) - \Phi(te^{\lambda N})| \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \},
 \end{aligned}$$

from which we deduce that

$$|\Phi(t) - \Phi(te^{\lambda N})| \leq \Phi(t) \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \}.$$

By using Lemma 9.3 with $te^{\lambda N}$ in place of t , we have

$$\begin{aligned}
 |I_1^+ - I_2^+| &\ll e^{-c_4(\log k)/\log_2 k} \frac{(1 + e^{\lambda \kappa_{te^{\lambda N}}})^{2N} \log T_k}{(e^{\gamma} te^{\lambda N})^{2\kappa_{te^{\lambda N}}}} \\
 &\quad + \frac{E(\kappa_{te^{\lambda N}}) + e^{-c_4(\log k)/\log_2 k}}{N(e^{\gamma} te^{\lambda N})^{2\kappa_{te^{\lambda N}}}} \left(\frac{1 + e^{\lambda \kappa_{te^{\lambda N}}}}{\lambda T_k} \right)^{2N}.
 \end{aligned}$$

On the other hand, by using Theorem 3 and (1.25), it is easy to see that there is a positive constant c such that

$$\Phi(te^{\lambda N}) \asymp \Phi(t) \sim \frac{E(\kappa_t)}{\kappa_t \sqrt{2\pi\sigma_2}(e^{\gamma t})^{2\kappa_t}} \gg e^{-c_8 e^t/t} \gg e^{-c_9(\log k)/[(\log_2 k)^{7/2} \log_3 k]}$$

for $t \leq T(k)$. Thanks to Lemma 4.5, the previous estimate can be written as

$$(9.14) \quad |I_1^+ - I_2^+| \ll \Phi(t) \frac{1}{N} \left(\frac{\kappa_{te^{\lambda N}}}{\log \kappa_{te^{\lambda N}}} \right)^{1/2} \left(\frac{1 + e^{\lambda \kappa_{te^{\lambda N}}}}{\lambda T_k} \right)^{2N} \ll \frac{\Phi(t)}{(\log k)^A}.$$

Similarly we can prove (even more easily)

$$(9.15) \quad |I_1 - I_2| \ll \Phi(t)/(\log k)^A.$$

Inserting (9.12) and (9.16) into (9.9) and (9.13) and (9.15) into (9.10), we obtain

$$\tilde{F}_k(t) \leq \Phi(t) \{1 + e^{t-T(k)-C}(t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}} + (\log k)^{-A})\}$$

and

$$\tilde{F}_k(t) \geq \Phi(t) \{1 - e^{t-T(k)-C}(t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}} + (\log k)^{-A})\}.$$

This implies the first asymptotic formula of (1.13) by taking $\eta = \frac{1}{200}$ and $\delta = \frac{1}{5}$.

The second can be established similarly. This completes the proof of Theorem 2. \square

§ 10. Proof of Theorem 1

The formula (1.9) is an immediate consequence of Theorem 2 and (1.25).

Taking $t = T(k)$ in (1.9), we find that

$$(10.1) \quad e^{-c'_1(\log k)/\{(\log_2 k)^{7/2} \log_3 k\}} \ll \tilde{F}_k(T(k)) \ll e^{-c'_2(\log k)/\{(\log_2 k)^{7/2} \log_3 k\}},$$

where c'_1 and c'_2 are two positive constants. Clearly (10.1) and (1.8) imply (1.11).

The related results on $\tilde{G}_k(t)$ and $G_k(T(k))$ can be proved similarly. This completes the proof of Theorem 1. \square

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